



Equilibria of Infinitely Repeated Games

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Strategy Space



- What is a pure strategy in an infinitely-repeated game?
 - a choice of action at every decision point
 - here, that means an action at every stage game
 - ...which is an infinite number of actions!
- Some famous strategies (repeated PD):
 - **Tit-for-tat**: Start out cooperating. If the opponent defected, defect in the next round. Then go back to cooperation.
 - **Trigger**: Start out cooperating. If the opponent ever defects, defect forever.

Nash Equilibria



- With an infinite number of pure strategies, what can we say about Nash equilibria?
 - we **won't** be able to construct an induced normal form and then appeal to Nash's theorem to say that an equilibrium exists
 - Nash's theorem only applies to finite games
- Furthermore, with an infinite number of strategies, there could be an infinite number of pure-strategy equilibria!
- We can characterize a set of **payoffs** that are achievable under equilibrium, without having to enumerate the equilibria.

Definitions

- Consider any n -player game $G = (N, A, u)$ and any payoff vector $r = (r_1, r_2, \dots, r_n)$.
- Let $v_i = \min_{s_{-i} \in S_{-i}} \max_{s_i \in S_i} u_i(s_{-i}, s_i)$.
 - i 's **minmax value**: the amount of utility i can get when $-i$ play a minmax strategy against him

Definition

A payoff profile r is **enforceable** if $r_i \geq v_i$.

Definition

A payoff profile r is **feasible** if there exist rational, non-negative values α_a such that for all i , we can express r_i as $\sum_{a \in A} \alpha_a u_i(a)$, with $\sum_{a \in A} \alpha_a = 1$.

- **feasible**: a convex, rational combination of the outcomes in G .



Folk Theorem



Theorem (Folk Theorem)

Consider any n -player game G and any payoff vector (r_1, r_2, \dots, r_n) .

1. If r is the payoff in any Nash equilibrium of the infinitely repeated G with average rewards, then for each player i , r_i is enforceable.
2. If r is both feasible and enforceable, then r is the payoff in some Nash equilibrium of the infinitely repeated G with average rewards.

Folk Theorem (Part I)

Payoff in Nash \Rightarrow enforceable

Part I: Suppose r is not enforceable, i.e. $r_i < v_i$ for some i .



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Folk Theorem (Part 2)

Feasible and enforceable \Rightarrow Nash

Part 2: Since r is a feasible payoff profile and the α 's are rational, we can write it as $r_i = \sum_{a \in A} \left(\frac{\beta_a}{\gamma} \right) u_i(a)$, where β_a and γ are non-negative integers and $\gamma = \sum_{a \in A} \beta_a$.



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We're going to construct a strategy profile that will cycle through all outcomes $a \in A$ of G with cycles of length γ , each cycle repeating action a exactly β_a times. Let (a^t) be such a sequence of outcomes.



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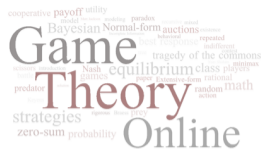
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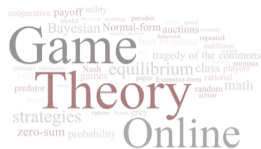
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First observe that if everybody plays according to s_i , then, by construction, player i receives average payoff of r_i (look at averages over periods of length γ). Second, this strategy profile is a Nash equilibrium. Suppose everybody plays according to s_i , and player j deviates at some point.

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